Direct Sum of Representations

Vector spaces $V_1, \ldots, V_n$

external direct sum $V = \underbrace{V_1 \oplus \cdots \oplus V_n}$

underlying set $= \{(v_1, \ldots, v_n) \mid v_k \in V\}$

obvious $+$, scalar mult.

e.g. $\mathbb{C}^n = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n}$

$\varphi^{(k)} : G \to GL(V_k)$, $\quad k = 1, \ldots, n$

$\rightsquigarrow$ (external) direct sum representation

$$\varphi : G \to GL(V_1 \oplus \cdots \oplus V_n)$$

$$\varphi_g\big((v_1, \ldots, v_n)\big) = \big(\varphi_g^{(1)}(v_1), \ldots, \varphi_g^{(n)}(v_n)\big)$$

$g \in G$

<u>Notation</u>: $\underset{=}{\varphi} = \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$

Example: $V_1 = \mathbb{C}^{n_1}$, $V_2 = \mathbb{C}^{n_2}$

$$V_1 \oplus V_2 \cong \mathbb{C}^{n_1+n_2}$$

given $\varphi^{(j)} : G \to GL_{n_j}(\mathbb{C})$, $j = 1,2$

$$\Rightarrow \quad \varphi = \varphi^{(1)} \oplus \varphi^{(2)} : G \to GL_{n_1+n_2}(\mathbb{C})$$

$$\varphi_g = \left[\begin{array}{c|c} \varphi^{(1)}_g & 0 \\ \hline 0 & \varphi^{(2)}_g \end{array}\right]$$

---

internal direct sum

$V \geq V_1, \ldots, V_n$ subspaces

such that $V_1 \oplus \cdots \oplus V_n \xrightarrow{\sim} V$

$(v_1, \ldots, v_n) \longmapsto v_1 + \cdots + v_n$

[e.g. $n=2$: $V_1 \cap V_2 = \{0\}$, $V_1 + V_2 = V$]

internal direct sum of representations:

$\varphi: G \longrightarrow GL(V)$,

$V_1, \dots, V_n \leq V$ invariant subspaces

such that $V = V_1 \oplus \cdots \oplus V_n$

say φ is an internal direct sum

of the subreps. $(\varphi|_{V_k})$, $k = 1, \dots, n$

$$\Rightarrow \quad \varphi \sim (\varphi|_{V_1}) \oplus \cdots \oplus (\varphi|_{V_n}).$$

$\varphi: G \to GL(V)$ is **decomposable**

if it is a direct sum of two proper non-trivial subrepresentations

i.e., $V_1, V_2 \leq V$ invariant subspaces

$V_1 \neq 0$, $V_2 \neq 0$, $V_1 \cap V_2 = 0$

$V_1 + V_2 = V$

$\Rightarrow \varphi \sim (\varphi|_{V_1}) \oplus (\varphi|_{V_2})$

$\varphi: G \to GL(V)$ is **completely reducible.**

if $\exists$ $\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$

s.t each $\varphi^{(k)}$ is irreducible